Lattice QED and Universality of the Axial Anomaly [*]

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We give a perturbative proof that U(1) lattice gauge theories generate the axial anomaly in the continuum limit under very general conditions on the lattice Dirac operator. These conditions are locality, gauge covariance and the absense of species doubling. They hold for Wilson fermions as well as for realizations of the Dirac operator that satisfy the Ginsparg-Wilson relation. The proof is based on the lattice power counting theorem. The results generalize to non-abelian gauge theories.

1. Introduction

In the continuum formulation of QED (or QCD) it is well known that all known gauge invariant regularization schemes yield the ABJ axial anomaly. A non-perturbative regularization is provided by the lattice. Any candidate for a lattice discretization of the QED action should reproduce the axial anomaly in the continuum limit. A naive symmetric discretization of the continuum Dirac operator fails to generate the anomaly, but is also plagued with the fermion doubling problem. The doubling problem can be eliminated by introducing an explicit, naively irrelevant, chiral symmetry breaking Wilson term in the action. This circumvents the Nielsen Ninomyia theorem [1] and leads to the correct anomaly in the continuum limit [2,3].

The question arises whether this is a particular property of the action chosen, or whether the anomaly will be correctly reproduced by any lattice regularization of the action satisfying some very general conditions. In fact, as we will show, any lattice action which is i) gauge invariant, ii) possesses the correct continuum limit, iii) is local in some generalized sense, and iv) is free of doublers, will necessarily reproduce the axial anomaly in the continuum limit. An important ingredient which will be needed for the proof is the general form of the lattice axial vector Ward identity. Its precise structure, which will depend on the particular discretization of the action cho-sen, need not be known, but only very general properties thereof. Our general proof will include the case of Ginsparg-Wilson fermions.

2. Ward Identity

Consider the following general form for the fermionic contribution to the lattice action

$$S_F[U, \psi, \bar{\psi}] = \sum_{x,y} \bar{\psi}(x)(D_U(x,y) + m)\psi(y) . \quad (1)$$

Here U stands for the collection of link variables, D_U is the Dirac operator, and $\sum_x \equiv \sum_n a^4$, where n label the lattice sites. D_U can be decomposed into a part anticommuting with γ_5, which in the continuum limit takes the form $\gamma \cdot \partial$, and a term commuting with γ_5, which vanishes in the continuum limit. Hence the general structure of the lattice axial vector Ward identity in an external gauge field will be of the form

$$\langle \partial_\mu^* j_\mu^5(x) \rangle_U = 2m \langle j^5(x) \rangle_U + \langle \Delta(x) \rangle_U \quad (2)$$

where $\partial_\mu^*(\partial_\mu)$ stands for the dimensioned left (right) lattice derivative. $j^5(x) = \bar{\psi}(x)\gamma_5\psi(x)$, and $j_\mu^5(x)$ has the correct continuum limit. $\Delta(x)$ is an "irrelevant" operator which vanishes in the naive continuum limit. For Wilson fermions j_μ^5 and Δ have been given in [2]. Furthermore, it was shown in [2,3], that the anomaly is generated in the continuum limit by the naively "irrelevant" contribution $< \Delta(x) >$. For Ginsparg-Wilson fermions the Dirac operator in (1) satisfies the Ginsparg Wilson relation [4]

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D ,$$

where a is the lattice spacing. The action possesses an exact global axial symmetry [5] for $m = 0$ under the transformations

$$\delta\psi(x) = i\omega\gamma_5[(1 - \frac{a}{2}D)\psi](x) \tag{3}$$

$$\delta\bar{\psi}(x) = i\omega[\bar{\psi}(1 - \frac{a}{2}D)](x)\gamma_5 . \tag{4}$$

According to the lattice Poincare Lemma [6] there exists an axial vector current associated with this symmetry. Under a local transformation with ω replaced by $\omega(x)$, the variation of the contribution to the fermionic action involving the Dirac operator in (1) can be written in the form $\delta S_D = -i\sum_x \omega(x)\partial^*_\mu j^5_\mu(x)$. Taking into account the variation of the fermionic measure one then finds that the axial Ward identity is of the form (2), where the irrelevant Δ operator is given by

$$\Delta_{GW}(x) = a(\bar{\psi}D)(x)\gamma_5(D\psi)(x) .$$

Consider now the Ward identity (2) in momentum space. Define $< j^5_\mu(q) >_U$ by

$$< j^5_\mu(x) >_U = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4q}{(2\pi)^4} < j^5_\mu(q) >_U e^{i(q\cdot x + a\hat{\mu}/2)}$$

Expanding $U_\mu(x) = exp[igaA_\mu(x)]$ in the potentials, $< j^5_\mu(q) >_U$ will have the form

$$< j^5_\mu(q) >_U = \sum_n \frac{1}{n!} \sum_{\{\nu_i\}} \int_{\{k_i\}} \delta(\sum_i k_i - q)$$
$$\times \Gamma^{5\mu}_{\nu_1\cdots\nu_n}(k_1,\cdots,k_n)A_{\nu_1}(k_1)\cdots A_{\nu_n}(k_n) ,$$

with analogous expressions for $< j^5(q) >_U$ and $< \Delta(q) >_U$. The Ward identity in momentum space then reads

$$\tilde{q}_\mu\Gamma^{5\mu}_{\{\nu_i\}}(\{k_i\}) = 2m\Gamma^5_{\{\nu_i\}}(\{k_i\}) + \Gamma^{(\Delta)}_{\{\nu_i\}}(\{k_i\}) , \tag{5}$$

where $\tilde{q}_\mu = \frac{2}{a}\sin\frac{q_\mu a}{2}$.

3. Theorem

We now state the central theorem of this paper and then give the proof:

Any lattice discretization of the QED action with the following properties: a) S has the correct continuum limit; b) S is gauge invariant; c) The Dirac Operator is local and d) The free Dirac operator $D^{(0)}(p)$ is free of doublers, reproduces the axial anomaly in the continuum limit.

As we shall see below, the explicit form of the axial vector current and of the irrelevant operator $\Delta(x)$ is not required. Let us consider a) to c) in turn:

1) *S has the correct continuum limit*: This ensures that j^5_μ posesses the correct naive continuum limit.

2) *Gauge invariance*: Gauge invariance tells us that if $\mathcal{O}(\psi,\bar{\psi},A)$ is a gauge invariant operator, then its external field expectation value satisfies

$$< \mathcal{O}(\psi,\bar{\psi},A^\omega) >_{A^\omega} = < \mathcal{O}(\psi,\bar{\psi},A) >_A ,$$

where $A^\omega_\mu(x) = A_\mu(x) + \partial_\mu\omega(x)$. It follows that

$$\sum_\nu \partial^*_\nu \frac{\partial}{\partial A_\nu(z)} < \mathcal{O}(\psi,\bar{\psi},A) >_A = 0 ,$$

or in momentum space

$$\sum_{\nu_i}(\tilde{k}_i)_{\nu_i}\Gamma^{(\mathcal{O})}_{\nu_1\cdots\nu_n}(k_1,\cdots,k_n) = 0 . \tag{6}$$

3) *Locality of the Dirac Operator*: The Dirac operator $D_U(x,y)$ can be formally expanded in a power series in the gauge potentials:

$$D_U(x,y) = \sum_{n,\mu_i,z_i} \frac{1}{n!}D^{(n)}_{\mu_1\cdots\mu_n}(x,y|z_1\cdots z_n)$$
$$\times A_{\mu_1}(z_1)\cdots A_{\mu_n}(z_n) .$$

The coefficient functions $D^{(n)}_{\mu_1\cdots\mu_n}$ are assumed to vanish exponentially fast with the separation between any pair of lattice sites with a decay constant of the order of the lattice spacing (this also holds for Ginsparg Wilson fermions for sufficiently smooth gauge field configurations [7]. As a consequence $\Gamma^{5\mu}_{\nu_1\cdots\nu_n}(\{k_i\})$, $\Gamma^5_{\nu_1\cdots\nu_n}(\{k_i\})$, and $\Gamma^{(\Delta)}_{\nu_1\cdots\nu_n}(\{k_i\})$ can be Taylor expanded around zero momenta. It then follows from (6) that

$$\mathcal{T}_1\Gamma_{\nu_1,\cdots,\nu_n}(k_1,\cdots,k_n) = 0 , \tag{7}$$

where $\mathcal{T}_n$ denote the Taylor expansion up to order n in the momenta

4) *The free propagator $D^{-1}_{(0)}(p)$ is free of doublers.* As a consequence the lattice power counting rules [8] and the Reisz theorem [9] applies. This theorem allows one to take the naive continuum limit of a lattice integral, if all the lattice degree of divergences of the integrand, including the measure, are negative.

Proof of the Theorem

In the following we only consider the axial vector Ward identity for two external photons, since it is the only one which is anomalous. It has the form

$$\tilde{q}_\mu \Gamma^{5\mu}_{\nu_1\nu_2}(\{k_i\}) = 2m\Gamma^5_{\nu_1\nu_2}(\{k_i\}) + \Gamma^{(\Delta)}_{\nu_1\nu_2}(\{k_i\}) \ , \quad (8)$$

where $i = 1, 2$. By power counting $degr\Gamma^{5\mu} = degr\Gamma^5 = 1$, while the irrelevant contribution has degr $\Gamma^{(\Delta)} \le 2$. Here $degrF$ denotes the lattice divergence degree of F [8].

We first show that all the vertex functions possess a continuum limit. Consider e.g. $\Gamma^{5\mu}$, and write it in the form

$$\Gamma^{5\mu}_{\nu_1\nu_2} = (1 - \mathcal{T}_1)\Gamma^{5\mu}_{\nu_1\nu_2} + \mathcal{T}_1\Gamma^{5\mu}_{\nu_1\nu_2} \qquad (9)$$

The first term on the RHS has an $degr < 0$. Hence making use of the Reisz theorem, its continuum limit is given by the Taylor subracted continuum one loop integral with a $\gamma_\mu\gamma_5$ insertion. This is just the usual continuum triangle graph, Taylor subtracted to first order in the momenta. The second term on the RHS of (9) vanishes because of gauge invariance (7). The $a \to 0$ limit of $\Gamma^5_{\nu_1\nu_2}$ can be calculated in a similar way. Hence we immediately arrive at the result that the continuum limit of $\Gamma^{(\Delta)}_{\nu_1\nu_2}$ in (8) is universal, and given in terms of convergent continuum Feynman integrals. This limit can be easily computed. To this effect decompose $\Gamma^{(\Delta)}_{\nu_1\nu_2}$ as follows:

$$\Gamma^{(\Delta)}_{\nu_1\nu_2} = (1 - \mathcal{T}_2)\Gamma^{(\Delta)}_{\nu_1\nu_2} + \mathcal{T}_1\Gamma^{(\Delta)}_{\nu_1\nu_2} + (\mathcal{T}_2 - \mathcal{T}_1)\Gamma^{(\Delta)}_{\nu_1\nu_2}$$

where $\mathcal{T}_2$ denotes Taylor subtraction to second order in the momenta. Because of gauge invariance (7) the second term on the RHS vanishes. According to the Reisz theorem [9] the first term vanishes for $a \to 0$ since $degr[\Gamma^{(\Delta)}] \le 2$, and $\Gamma^{(\Delta)}$ vanishes in the naive continuum limit. Hence we conclude that

$$\lim_{a\to 0} \Gamma^{(\Delta)}_{\nu_1\nu_2}(k_1, k_2) = \lim_{a\to 0}(\mathcal{T}_2 - \mathcal{T}_1)\Gamma^{(\Delta)}_{\nu_1\nu_2}(k_1, k_2)$$

We now make use of the Ward identity (8) to obtain a simple expression for the RHS. Thus applying the $\mathcal{T}_2 - \mathcal{T}_1$ operation to both sides of (8), making use of gauge invariance (7) and of the Reisz theorem, one readily finds that

$$\lim_{a\to 0} \Gamma^{(\Delta)}_{\nu_1\nu_2} = 2m \int \frac{d^4\ell}{(2\pi)^4} \ (\mathcal{T}_1 - \mathcal{T}_2)I^5_{\nu_1\nu_2}(\ell, k_1, k_2)$$

where $I^5_{\nu_1\nu_2}(\ell, k_1, k_2)$ is the integrand of the continuum Feynman integral for the triangle graph with a γ_5 insertion. Hence the anomalous contribution reads:

$$\mathcal{A}_{\mu\nu} = -16e^2\epsilon_{\mu\nu\sigma\rho}k_{1\sigma}k_{2\rho} \int \frac{d^4\ell}{(2\pi)^4} \ \frac{m^2}{(\ell^2 + m^2)^3}$$

$$= -\frac{1}{2\pi^2}\sum_{\rho,\sigma}\epsilon_{\mu\nu\rho\sigma}k_{1\rho}k_{2\sigma}$$

We have thus shown that, irrespective of the detailed structure of the lattice current and of the irrelevant operator Δ in the Ward identity (8), the correct anomaly is generated by the irrelevant Δ-contribution in the continuum limit. This limit is universal and given by the second order Taylor term in the expansion of the continuum triangle graph with a γ_5 vertex insertion. This expression has precisely the same form as that obtained in the continuum formulation using the Pauli-Villars regularization scheme, except that here the Pauli Villars mass is replaced by the fermion mass. The integral is however independent of the mass. Furthermore we have seen that all terms in the Ward identity (8) posssess a continuum limit. The limit of the first two terms is however *not* given by the naive limit of the triangle graphs, but actually by their BPHZ subtracted form.

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